JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **35**, No. 1, February 2022
http://dx.doi.org/10.14403/jcms.2022.35.1.53

STUDIES ON MONOTONE ITERATIVE TECHNIQUE FOR NONLINEAR SYSTEM OF INITIAL VALUE PROBLEMS

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Abstract. Nonlinear system of initial value problems involving R-L fractional derivative is studied. Monotone iterative technique coupled with lower and upper solutions is developed for the problem. It is successfully applied to study qualitative properties of solutions of nonlinear system of initial value problem when the function on the right hand side is nondecreasing.

1. Introduction

Fractional differential equations (FDEs) arise in many scientific disciplines as the mathematical modeling of system and processes in the fields of chemistry, physics, electrodynamics of complex medium, aerodynamics, polymer rheology [9, 13, 27, 29]. Most of the researchers are attracted towards fractional differential equations as many phenomena in various branches of science and engineering are modeled. Many applications are found in control systems, visco-elasticity, electrochemistry, pharmacokinetics,food science etc.[10, 14, 27, 29]. Significant contributions by researchers have been recorded in the monograph of Kilbas et al.[9]. There are some good methods for studying fractional differential equations such as power series method, monotone method, compositional method and transform method [2, 6, 17, 25, 27, 28, 30]. The monotone iterative technique[3] is very useful for the investigation of theoretical as well as constructive results in the sector. McRae developed monotone method for Riemann-Liouville fractional differential equations with initial conditions and studied the qualitative properties of solutions

Received Agust 03, 2021; Accepted September 28, 2021.
2010 Mathematics Subject Classification: 34A12, 34C60, 34A45.
Key words and phrases: monotone iterative technique, lower-upper solutions, initial value problems, fractional differential equations.

of initial value problem (IVP) in [1, 15]. In 2011, Denton et al.[4] developed monotone method combined with the method of coupled upper and lower solutions for finite systems of fractional differential equations with initial conditions. Existence of solutions of Riemann-Liouville fractional differential equations and uniqueness of solutions proved by Lakshmikantham and Vatsala [11, 12]. Existence and uniqueness of solution of Riemann-Liouville fractional differential equations with integral boundary conditions is also obtained by Nanware et al.[17, 18, 20, 23]. Moreover, Nanware et al.[8, 16, 19, 20, 21, 22, 24, 26] developed monotone method for system of fractional differential equations with various conditions and successfully applied to study qualitative properties of solutions. Recently, Wei et al.[31, 32] and Nanware et. al.[7], developed monotone iterative technique to study existence and uniqueness results for initial value problems and periodic boundary value problems involving Riemann- Liouville sequential fractional derivative and technique. In this paper, we develop monotone iterative technique to study existence and uniqueness of solution of the following nonlinear system with initial conditions:

$$(D^{2q}u_i)(t) = f_i(t,\, u_1,\, u_2, D^q u_1, D^q u_2), t \in (0, T] \tag{1.1}$$

$$t^{1-q}u_i(t)|_{t=0} = u_i^0, t^{1-q}(D^q u_i)(t)|_{t=0} = u_i^1,\; i = 1,\; 2. \tag{1.2}$$

where $0 < T < \infty$, u_i^0, u_i^1 are constants and $f_i \in C([0,\, T] \times \mathbb{R}^4),\; i = 1, 2,$ is quasimonotone nondecreasing, D^q is the standard Riemann- Liouville fractional derivative of order $0 < q \leq 1$. We organize the paper as follows. In section 2, preliminary definitions and some basic results are considered. Some important lemmas and comparison results are also given. In section 3, we develop monotone technique for system of IVP with Riemann-Liouville fractional differential equation. Existence and uniqueness of solution of coupled system of IVP are obtained.

2. Preliminaries

In this section, we deduce some preliminary results that will be used in the next section to attain existence and uniqueness results for the nonlinear system of initial value problem (1.1)- (1.2). Assume that $J = [0,\, T] \subset \mathbb{R}$ is a compact interval and

$$f_i(t,\, u_1(t),\, u_2(t),\, D^q u_1(t),\, D^q u_2(t)) \in C(J \times \mathbb{R}^4,\, \mathbb{R}),\;\; i = 1,\; 2$$

is quasimonotone nondecreasing. Define the following classes:

$$C([0,\,T]) = \big\{u_i | u_i(t) \text{ is continuous on } [0,\,T],\ \|u_i\|_C = \max_{t\in[0,\,T]} |u_i(t)|\big\},$$

$$C_{1-q}([0,\,T]) = \big\{u_i \in C([0,\,T]) : t^{1-q}u_i(t) \in C([0,\,T]),$$
$$|u_i\|_{C_{1-q}} = \|t^{1-q}u_i(t)\|_C\big\},$$

$$C^q_{1-q}([0,\,T]) = \{u_i \in C_{1-q}([0,\,T]) : t^{1-q}D^q u_i(t) \in C([0,\,T])\}.$$

Definition 2.1. A function $f_i(t,\,u_1(t),\,u_2(t),\,D^q u_1(t),\,D^q u_2(t)) \in C(J\times\mathbb{R}^4,\,\mathbb{R})$, $i = 1,2$, $J = [0,\,T]$ is said to be quasimonotone nondecreasing (nonincreasing) if for each i, $u_i \le v_i$ and $u_j = v_j$, $i \ne j$, then

$$f_i(t,\,u_1,\,u_2,\,D^q u_1(t),\,D^q u_2(t)) \le f_i(t,\,v_1,\,v_2,\,D^q v_1(t),\,D^q v_2(t))$$
$$\left(f_i(t,\,u_1,\,u_2,\,D^q u_1(t),\,D^q u_2(t)) \ge f_i(t,\,v_1,\,v_2,\,D^q v_1(t),\,D^q v_2(t))\right).$$

Definition 2.2. A function $v_i^0 = (v_1^0,\,v_2^0) \in C^q_{1-q}([0,\,T])$ is called a lower solution of IVP(1.1)- (1.2) if it satisfies

$$(D^{2q}v_i^0)(t) \le f_i(t,\,v_1(t),\,v_2(t),\,D^q v_1(t),\,D^q v_2(t)), t\in(0,\,T]$$
$$t^{1-q}v_i^0(t)|_{t=0} \le v_i^0,\ t^{1-q}D^q v_i^0(t)|_{t=0} \le v_i^1.$$

Definition 2.3. A function $w_i^0 = (w_1^0,\,w_2^0) \in C^q_{1-q}([0,\,T])$ is called a upper solution of IVP(1.1)-(1.2), if it satisfies

$$(D^{2q}w_i^0)(t) \ge f_i(t,\,w_1(t),\,w_2(t),\,D^q w_1(t),\,D^q w_2(t)),\ t\in(0,\,T]$$
$$t^{1-q}w_i^0(t)|_{t=0} \ge w_i^0,\ t^{1-q}D^q w_i^0(t)|_{t=0} \ge w_i^1.$$

Definition 2.4. The sector denoted by Ω is defined as

$$\Omega = \left[v_i^0,\,w_i^0\right] = \big\{u_i \in C^q_{1-q}([0,\,T]) : v_i^0 \le u_i \le w_i^0,\ t\in[0,\,T];$$
$$t^{1-q}v_i^0(t)|_{t=0} \le t^{1-q}u_i(t)|_{t=0} \le t^{1-q}w_i^0(t)|_{t=0},$$
$$t^{1-q}D^q v_i^0(t)|_{t=0} \le t^{1-q}D^q u_i(t)|_{t=0} \le t^{1-q}D^q w_i^0(t)|_{t=0}\big\}.$$

Definition 2.5. Let $f_i : J\times\mathbb{R}^4 \to \mathbb{R}$ be a real valued continuous function. We say that $f_i(t,u_1(t),u_2(t),D^q u_1(t),D^q u_2(t))$ satisfies one sided Lipschitz condition, if there exist constants $M_i,\ N_i \in \mathbb{R}$, $N_i^2 > 4M_i$, such that

$$f_i(t,w_1,w_2,D^q w_1,D^q w_2) - f_i(t,v_1,v_2,D^q v_1,D^q v_2) \ge -N_i(D^q w_i - D^q v_i) - M_i(w_i - v_i), \tag{2.1}$$

$v_i^0 \le v_i \le w_i \le w_i^0, i = 1,2.$

Further to ensure the uniqueness of solution of IVP (1.1) − (1.2), there exist constants $M_i, N_i \in \mathbb{R}$, $N_i^2 > 4M_i$, such that

$$f_i(t, w_1, w_2, D^q w_1, D^q w_2) - f_i(t, v_1, v_2, D^q v_1, D^q v_2) \le N_i(D^q w_i - D^q v_i) + M_i(w_i - v_i), \tag{2.2}$$

$v_i^0 \le v_i \le w_i \le w_i^0$.

From conditions (2.1) and (2.2), we conclude that the function f_i satisfies Lipschitz condition if there exists constants $N_i,\ M_i \ge 0,\ N_i^2 > 4M_i$ such that

$$|f_i(t, w_1, w_2, D^q w_1, D^q w_2) - f_i(t, v_1, v_2, D^q v_1, D^q v_2)| \le N_i|D^q w_i - D^q v_i| + M_i|w_i - v_i|. \tag{2.3}$$

Now, we consider the following result for the linear fractional initial value problem to obtain existence and uniqueness results of solution of the IVP (1.1) − (1.2).

LEMMA 2.6. [9] *Suppose that $u(t) \in C_{1-q}([0, T])$, then the linear initial value problem*

$$D^q u(t) + Mu(t) = \sigma(t),\ t \in (0, T], \quad t^{1-q}u(t)|_{t=0} = u_0,$$

where $M \in \mathbb{R}$ and $\sigma(t) \in C_{1-q}([0, T])$, has the following integral representation of solution

$$u(t) = \Gamma(q)u_0 e_q(-Mt) + [e_q(-Mx) * \sigma(x)](t),$$

where

$$(g * f)(t) = \int_0^t g(t - x)f(x)\, dx,$$

and

$$e_q(\lambda z) = z^{q-1}E_{q,\,q}(\lambda z^q) = z^{q-1}\sum_{k=0}^{\infty} \lambda^k \frac{z^{qk}}{\Gamma((k+1)q)},$$

where $E_{q,\,q} = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{\Gamma((k+1)q)}$, is Mittag-Leffler function of two parameter.

LEMMA 2.7. [31] *Suppose that $u(t) \in C_{1-q}^q([0, T])$ then the linear initial value problem*

$$\begin{aligned} &(D^{2q}u)(t) + N(D^q u)(t) + Mu(t) = \sigma(t), t \in (0, T] \\ &t^{1-q}u(t)|_{t=0} = u_0,\ t^{1-q}D^q u(t)|_{t=0} = u_1, \end{aligned} \tag{2.4}$$

where $M,\, N \in \mathbb{R}$ *are constants,* $N^2 > 4M$ *and* $\sigma(t) \in C_{1-q}([0,\, T])$*, has the following representation of solution*

$$u(t) = \Gamma(q)u_0 e_q(\lambda_2 t) + \Gamma(q)(u_1 - \lambda_2 u_0)[e_q(\lambda_2 x) * e_q(\lambda_1 x)](t) \\ + [e_q(\lambda_2 x) * e_q(\lambda_1 x) * \sigma(x)](t), \tag{2.5}$$

where

$$\lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2},\ \lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2} \leq 0.$$

Lemma 2.8. [31]

$$[e_q(\lambda_2 x) * e_q(\lambda_1 x)](t) = [e_q(\lambda_1 x) * e_q(\lambda_2 x)](t) = \frac{1}{\lambda_1 - \lambda_2}[e_q(\lambda_1 x) - e_q(\lambda_2 x)](t).$$

Lemma 2.9. *Comparison result* [31] *If* $u(t) \in C_{1-q}([0,\, T])$ *and satisfies the relation*

$$D^q u(t) + M u(t) \geq 0,\ t \in (0,\, T], t^{1-q} u(t)|_{t=0} \geq 0,$$

where $M \in \mathbb{R}$ *is a constant. Then* $u(t) \geq 0,\ t \in (0,\, T]$.

Lemma 2.10. *Comparison result* [31] *If* $u(t) \in C^q_{1-q}([0,\, T])$ *and satisfies the relation* $(D^{2q}u)(t) + N(D^q u)(t) + M u(t) = \sigma(t) \geq 0,\ t \in (0,\, T], t^{1-q}u(t)|_{t=0} = u_0 \geq 0,\ t^{1-q}D^q u(t)|_{t=0} = u_1 \geq 0,$ *where* $N,\, M \in \mathbb{R}$ *,* $N^2 > 4M$ *are constants such that* $\lambda_1 = \frac{-N+\sqrt{N^2-4M}}{2} \geq 0 > \lambda_2 = \frac{-N-\sqrt{N^2-4M}}{2}$*. Then* $u(t) \geq 0,\ t \in (0,\, T]$.

3. Main Results

In this section, we prove the existence and uniqueness theorem of solution for IVP (1.1) – (1.2).

Theorem 3.1. *Assume that*

(i) $v_i^0 = (v_1^0, v_2^0)$ *and* $w_i^0 = (w_1^0, w_2^0)$ *in* $C^q_{1-q}([0,\, T])$ *are ordered lower and upper solutions of IVP* (1.1) – (1.2) *respectively.*

(ii) $f_i \equiv f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_2) \in C(J \times \mathbb{R}^4, \mathbb{R})$*,* $J = [0,\, T]$ *satisfies one-sided Lipschitz condition,* $i = 1, 2$.

(iii) $f_i \equiv f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_2)$ *are quasi-monotone non-decreasing*

then there exist monotone sequences $\{v_i^n\}$ *and* $\{w_i^n\}$ *such that* $\{v_i^n\} \to v_i(t)$*,* $\{w_i^n\} \to w_i(t)$ *as* $n \to \infty$*, where* $v(t)$ *and* $w(t)$ *are minimal and* *maximal solutions on the ordered interval* $[v_i^0, w_i^0]$ *of IVP* (1.1) − (1.2) *and satisfy the monotone property*

$$v_i^0 \le v_i^1 \le v_i^2 \ldots \le v_i^n \le v_i \le w_i \le w_i^n \le \ldots \le w_i^1 \le w_i^0, \quad i = 1, 2.$$

Proof. For any $\eta_i(t) = (\eta_1, \eta_2) \in \Omega$. Consider the linear initial value problem

$$\begin{aligned}(D^{2q}u_i)(t) &= f_i(t,\, \eta_1,\, \eta_2,\, D^q\eta_1,\, D^q\eta_2) + N_i(D^q\eta_i - D^q u_i) + M_i(\eta_i - u_i)\\ &= \sigma(\eta_i)\end{aligned} \tag{3.1}$$
$$t^{1-q}u_i(t)|_{t=0} = u_i^0,\ t^{1-q}(D^q u_i)(t)|_{t=0} = u_i^1,\ i = 1, 2.$$

It is clear that, by Lemma 2.7 and 2.8, linear initial value problem (3.1) has exactly one solution $u_i \in C^q_{1-q}([0,\, T])$ and whose integral representation is as in (2.5). Now define

$$\begin{aligned}u_i(t) &= A[\eta_i,\, \mu]\\ &= \Gamma(q)u_i^0 e_q(\lambda_2^i t) + \Gamma(q)(u_i^1 - \lambda_2^i u_i^0)[e_q(\lambda_2^i x) * e_q(\lambda_1^i x)](t) +\\ &\qquad [e_q(\lambda_2^i x) * e_q(\lambda_1^i x) * \sigma(\eta_i)(x)](t),\end{aligned}$$

where

$$\lambda_1^i = \frac{-N_i + \sqrt{N_i^2 - 4M_i}}{2} \ge 0 > \lambda_2^i = \frac{-N_i - \sqrt{N_i^2 - 4M_i}}{2}.$$

For each $\eta_i(t) = (\eta_1, \eta_2)$ and $\mu_i(t) = (\mu_1, \mu_2)$ in Ω such that $v_i^0(0) \le \eta_i(t) \le \mu_i(t) \le w_i^0(0)$. We define an operator A from $[v_i^0,\, w_i^0]$ into $C^q_{1-q}([0,\, T])$ and η_i is solution of the IVP (1.1)-(1.2) if and only if $\eta_i = A[\eta_i, \mu]$ and μ_i is solution of the IVP (1.1)-(1.2) if and only if $\mu_i = A[\eta, \mu_i]$. First we prove that

(a) $v_i^0 \le A[v_i^0,\, w_i^0],\ w_i^0 \ge A[w_i^0,\, v_i^0]$
(b) A possesses the monotone property on the segment $\Omega = [v_i^0,\, w_i^0]$.

To prove (a), set $A[v_i^0,\, w_i^0] = v_i^1(t) = (v_1^1, v_2^1)$, where $v_i^1(t)$ is the unique solution of system (3.1) and set $p_i(t) = v_i^0(t) - v_i^1(t)$ with $\eta_i = v_i^0(t)$.

Observe that

$$
\begin{aligned}
D^{2q}p_i(t) &= D^{2q}v_i^0(t) - D^{2q}v_i^1(t)\\
&\le f_i(t, v_1^0, v_2^0, D^q v_1^0, D^q v_2^0) - f_i(t, v_1^0, v_2^0, D^q v_1^0, D^q v_2^0)\\
&\qquad\qquad - N_i(D^q v_i^0 - D^q v_i^1) - M_i(v_i^0 - v_i^1)\\
&= -N_i(D^q v_i^0 - D^q v_i^1) - M_i(v_i^0 - v_i^1)\\
&= -N_i(D^q p_i)(t) - M_i(p_i)(t).\\
Thus \quad D^{2q}p_i(t) &\le -N_i(D^q p_i)(t) - M_i(p_i)(t)\\
and \quad t^{1-q}p_i(t)|_{t=0} &= t^{1-q}v_i^0(t)|_{t=0} - t^{1-q}v_i^1(t)|_{t=0} \le 0\\
t^{1-q}(D^q p_i)(t)|_{t=0} &= t^{1-q}(D^q v_i^0)(t)|_{t=0} - t^{1-q}(D^q v^1)_i(t)|_{t=0} \le 0.
\end{aligned}
$$

By Lemma 2.10, we have $p_i(t) \le 0 \Rightarrow v_i^0(t) - v_i^1(t) \le 0 \Rightarrow v_i^0(t) \le v_i^1(t) = A[v_i^0, w_i^0]$. To prove that $w_i^0 \ge A[w_i^0, v_i^0]$, set $A[w_i^0, v_i^0] = w_i^1$, where w_i^1 is the unique solution of system (3.1). Set $p_i(t) = w_i^0 - w_i^1$ with $\eta_i = w_i^0(t)$.

$$
\begin{aligned}
D^{2q}p_i(t) &= D^{2q}w_i^0(t) - D^{2q}w_i^1(t)\\
&\ge f_i(t, w_1^0, w_2^0, D^q w_1^0, D^q w_2^0) - f_i(t, w_1^0, w_2^0, D^q w_1^0, D^q w_2^0)\\
&\qquad\qquad - N_i(D^q w_i^0 - D^q w_i^1) - M_i(w_i^0 - w_i^1)\\
&= -N_i(D^q w_i^0 - D^q w_i^1) - M_i(w_i^0 - w_i^1).\\
Thus \quad D^{2q}p_i(t) &\ge -N_i(D^q p_i) - M_i(p_i),\\
and \quad t^{1-q}p_i(t)|_{t=0} &= t^{1-q}w_i^0(t)|_{t=0} - t^{1-q}w_i^1(t)|_{t=0} \ge 0\\
t^{1-q}(D^q p_i)(t)|_{t=0} &= t^{1-q}(D^q w_i^0)(t)|_{t=0} - t^{1-q}(D^q w^1)_i(t)|_{t=0} \ge 0.
\end{aligned}
$$

By Lemma 2.10, we have $p_i(t) \ge 0 \Rightarrow w_i^0(t) - w_i^1(t) \ge 0 \Rightarrow w_i^0(t) \ge w_i^1(t) = A[w_i^0, v_i^0]$.
Now to prove (b), if $v_i^0 \le \eta_i \le \mu_i \le w_i^0$, then prove $A[\eta_i, \mu] \le A[\eta, \mu_i]$, where $A[\eta_i, \mu] = u_i = (u_i^1, u_i^2)$ and $A[\eta, \mu_i] = v_i = (v_i^1, v_i^2)$. Consider

$p_i(t) = u_i(t) - v_i(t)$ then observe that

$$\begin{aligned}
(D^{2q}p_i)(t) &= (D^{2q}u_i)(t) - (D^{2q}v_i)(t) \\
&= f_i(t, \eta_1, \eta_2, D^q\eta_1, D^q\eta_2) - f_i(t, \mu_1, \mu_2, D^q\mu_1, D^q\mu_2) + \\
&\qquad N_i(D^q\eta_i - D^q u_i) + M_i(\eta_i - u_i) - \\
&\qquad N_i(D^q\mu_i - D^q v_i) - M_i(\mu_i - v_i) \\
&\le N_i(D^q\mu_i - D^q\eta_i) + M_i(\mu_i - \eta_i) + N_i(D^q\eta_i - D^q u_i) + \\
&\qquad M_i(\eta_i - u_i) - N_i(D^q\mu_i - D^q v_i) - M_i(\mu_i - v_i) \\
&= N_i(D^q v_i - D^q u_i) + M_i(v_i - u_i).
\end{aligned}$$

Thus $(D^{2q}p_i)(t) \le -N_i(D^q p_i)(t) - M_i(p_i)(t)$

$$t^{1-q}p_i(t)|_{t=0} = t^{1-q}u_i(t)|_{t=0} - t^{1-q}v_i(t)|_{t=0} = u_i^0 - v_i^0 \le 0$$

$$t^{1-q}(D^q p_i)(t)|_{t=0} = t^{1-q}(D^q u_i)(t)|_{t=0} - t^{1-q}(D^q v_i)(t)|_{t=0} = u_i^1 - v_i^1 \le 0.$$

By Lemma 2.10, $p_i(t) \le 0 \Rightarrow u_i(t) \le v_i(t)$. Hence $A[\eta_i, \mu] \le A[\eta, \mu_i]$. Thus the operator A possesses the monotone property on $\Omega = [v_i^0, w_i^0]$. Define the sequences $\{v_i^n(t)\}$ and $\{w_i^n(t)\}$ by $v_i^n = A[v_i^{n-1}, w_i^{n-1}]$ and $w_i^n = A[w_i^{n-1}, v_i^{n-1}]$. Then, we obtain $v_i^0 \le v_i^1 \le v_i^2 \ldots \le v_i^n \le v_i \le w_i \le w_i^n \le \ldots \le w_i^1 \le w_i^0$. Let $P_i = \{v_i^n : n \in \mathbb{N}\}$, $Q_i = \{w_i^n : n \in \mathbb{N}\}$. We show that the set P_i, Q_i is relatively compact in $C_{1-q}^q([0, T])$. For any $\eta_i(t) \in \Omega$ by definition of lower and upper solutions and Lipschitz condition, we have

$$\begin{aligned}
(D^{2q}v_i^0)(t) + N_i(D^q v_i^0)(t) + M_i v_i^0(t) &\le f_i(t, v_1^0, v_2^0, D^q v_1^0, D^q v_2^0) + \\
&\qquad N_i(D^q v_i^0)(t) + M_i(v_i^0)(t) \\
&\le f_i(t, \eta_1, \eta_2, D^q\eta_1, D^q\eta_2) + N_i(D^q\eta_i)(t) + M_i(\eta_i)(t) \\
&\le f_i(t, w_1^0, w_2^0, D^q w_1^0, D^q w_2^0) + N_i(D^q w_i^0)(t) + M_i(w_i^0)(t) \\
&\le (D^{2q}w_i^0)(t) + N_i(D^q w_i^0)(t) + M_i w_i^0(t).
\end{aligned}$$

Let P_i, Ω in $C_{1-q}^q([0, T])$ be bounded sets. Furthermore

$$\sigma_i(\eta_i(t)) = f_i(t, \eta_1, \eta_2, D^q\eta_1, D^q\eta_2) + N_i(D^q\eta_i)(t) + M_i(\eta_i)(t) : \eta_i \in \Omega$$

is also bounded set. Hence, there exist a constant $L_i > 0$ such that

$$||\sigma_i(v_i^n)(t)|| = \max_{0 \le t \le T} |t^{1-q}\sigma_i(v_i^n)(t)| \le L_i$$

$$\Leftrightarrow |\sigma_i(v_i^n)(t)| \le L_i t^{1-q}, t \in (0, T]. \tag{3.2}$$

On the other hand $\{v_i^n : n \in \mathbb{N}\}$ satisfy

$$v_i^n = \Gamma(q)u_0 e_q(\lambda_2^i t) + \Gamma(q)(u_1^i - \lambda_2^i u_0^i)[e_q(\lambda_2^i x) * e_q(\lambda_1^i x)](t) \\ + [e_q(\lambda_2^i x) * e_q(\lambda_1^i x) * \sigma(v_i^{n-1})](t). \tag{3.3}$$

Let

$$G(\lambda_j^i,\, t) = t^{1-q}[e_q(\lambda_j^i t) * \sigma(v_i^{n-1})](t)], t \in (0,\, T], i = 1, 2.$$

Without loss of generality, assume that $0 \leq t_1 \leq t_2 \leq T$. Since $\lambda_2^i < 0 \leq \lambda_1^i$, we have

$$|G(\lambda_2^i,\, t_1) - G(\lambda_2^i,\, t_2)| \leq \frac{L_i\Gamma(q)}{|\lambda_1^i|}|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| + \\ \frac{2L_i\Gamma(q)}{\Gamma(2q)}(t_2 - t_1)^q, \tag{3.4}$$

$$|G(\lambda_1^i,\, t_1) - G(\lambda_1^i,\, t_2)| \leq (\frac{L_i\Gamma(q)}{|\lambda_1^i|} + \frac{L_i T^q}{q})|E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)| \\ + \frac{2L_i\Gamma(q)}{\Gamma(2q)}E_{q,q}(\lambda_1^i T^q)(t_2 - t_1)^q. \tag{3.5}$$

From $E_{q,q}(t) \in C([0,\, T])$ and $\forall \epsilon > 0\ \exists\ \delta = \delta(\epsilon)$, when $|t_2 - t_1| < \delta$, we have

$$|E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)| < \frac{\epsilon}{6L_i^1}, \tag{3.6}$$

$$|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| < \frac{\epsilon}{6L_i^2}, \tag{3.7}$$

$$(t_2 - t_1)^q < \frac{\epsilon}{6L_i^3}, \tag{3.8}$$

where

$$L_i^1 = max\left\{\frac{\Gamma(q)|(u_i^1 - \lambda_2^i u_i^0)\lambda_1^i|}{|\lambda_1^i - \lambda_2^i|},\ \frac{L_i}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|}[\Gamma^2(q) + \frac{|\lambda_1^i|T^q}{q}]\right\},$$

$$L_i^2 = max\Bigg\{\Gamma(q)|u_i^0|,\ \frac{\Gamma(q)|(u_i^1 - \lambda_2^i u_i^0)\lambda_1^i|}{|\lambda_1^i - \lambda_2^i|}, \\ \frac{L_i}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|}[\Gamma^2(q) + \frac{|\lambda_1^i|T^q}{q}]\Bigg\},$$

$$L_i^3 = \frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|}[1 + E_{q,q}(\lambda_1^i T^q)].$$

Using (3.4) to (3.8), we obtain

$$
\begin{aligned}
|t_1^{1-q}v_i^n(t_1) - t_2^{1-q}v_i^n(t_2)| = \Bigg| & \Gamma(q)u_i^0\left[t_1^{1-q}e_q(\lambda_2^i t_1) - t_2^{1-q}e_q(\lambda_2^i t_2)\right] \\
& + \Gamma(q)(u_i^1 - \lambda_2^i u_i^0)\Big[t_1^{1-q}e_q(\lambda_2^i t_1) * e_q(\lambda_1^i t_1) - \\
& \qquad t_2^{1-q}e_q(\lambda_2^i t_2) * e_q(\lambda_1^i t_2)\Big] \\
& + \Big[t_1^{1-q}e_q(\lambda_2^i t_1) * e_q(\lambda_1^i t_1) * \sigma(v_i^{n-1})(t_1) \\
& - t_2^{1-q}e_q(\lambda_2^i t_2) * e_q(\lambda_1^i t_2) * \sigma(v_i^{n-1})(t_2)\Big]\Bigg| \\
= \Bigg| & \Gamma(q)u_i^0[E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)] \\
& + \frac{\Gamma(q)(u_i^1 - \lambda_2^i u_i^0)}{\lambda_1^i - \lambda_2^i}\Big[(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)) + \\
& \qquad (E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q))\Big] \\
& + \frac{1}{\lambda_1^i - \lambda_2^i}\Big\{[t_1^{1-q}e_q(\lambda_2^i t_1) * \sigma(v_i^{n-1})(t_1) - \\
& \qquad t_2^{1-q}e_q(\lambda_2^i t_2) * \sigma(v_i^{n-1})(t_2)] \\
& + [t_2^{1-q}e_q(\lambda_1^i t_2) * \sigma(v_i^{n-1})(t_2) - t_1^{1-q}e_q(\lambda_1^i t_1) * \sigma(v_i^{n-1})(t_1)]\Big\}\Bigg|, \\
\leq \Gamma(q)\left|u_i^0\right| & |E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| \\
& + \frac{\Gamma(q)|u_i^1 - \lambda_2^i u_i^0|}{|\lambda_1^i - \lambda_2^i|}\Big[|(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q))| + \\
& \qquad |(E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)|\Big] \\
& + \frac{1}{|\lambda_1^i - \lambda_2^i|}|t_1^{1-q}e_q(\lambda_2^i t_1) * \sigma(v_i^{n-1})(t_1) - \\
& \qquad t_2^{1-q}e_q(\lambda_2^i t_2) * \sigma(v_i^{n-1})(t_2)| \\
& + \frac{1}{|\lambda_1^i - \lambda_2^i|}|t_1^{1-q}e_q(\lambda_1^i t_1) * \sigma(v_i^{n-1})(t_1) - \\
& \qquad t_2^{1-q}e_q(\lambda_1^i t_2) * \sigma(v_i^{n-1})(t_2)]|,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Gamma(q)|u_i^0| \left[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)|\right] \\
&\quad + \frac{\Gamma(q)|u_i^1 - \lambda_2^i u_i^0|}{|\lambda_1^i - \lambda_2^i|}\Bigg[|(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q))| + \\
&\qquad\qquad |(E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)|\Bigg] \\
&\quad + \frac{L_i\Gamma(q)}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|}\Bigg[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| + \\
&\qquad\qquad \frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|}(t_2 - t_1)^q\Bigg] \\
&\quad + \frac{1}{|\lambda_1^i - \lambda_2^i|}\left[\frac{L_i\Gamma(q)}{|\lambda_1^i|} + \frac{L_i T^q}{q}\right]\left[|E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)|\right] \\
&\quad + \frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|}E_{q,q}(\lambda_1^i T^q)(t_2 - t_1)^q,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Gamma(q)|u_i^0| \left[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)|\right] + \\
&\quad \frac{\Gamma(q)|u_i^1 - \lambda_2^i u_i^0|}{|\lambda_1^i - \lambda_2^i|}\Bigg[|(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q))| + \\
&\qquad\qquad |E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)|\Bigg] \\
&\quad + \frac{L_i\Gamma(q)}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|}\left[\Gamma(q) + (\Gamma(q) + \frac{\lambda_1^i T^q}{q})\right] \\
&\qquad \Bigg[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| + |E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)|\Bigg] \\
&\quad + \frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|}\left[1 + E_{q,q}(\lambda_1^i T^q)\right](t_2 - t_1)^q, \\
&< \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.
\end{aligned}
$$

Thus P_i is equi-continuous. Then by Ascoli- Arzela theorem, we conclude that P_i is relatively compact set of $C_{1-q}^q([0,T])$. Similarly we can show that Q_i is relatively compact set of $C_{1-q}^q([0,T])$. Therefore the sequences $\{v_i^n(t)\}$,$\{w_i^n(t)\}$ converges uniformly to $v_i(t), w_i(t)$ respectively on $[0,T]$. Then we have point-wise limits $\lim_{n\to\infty} v_i^n(t) = v_i(t), \lim_{n\to\infty} w_i^n(t) = w_i(t)$, $\underset{n\to\infty}{lim} D^q v_i^n(t) = D^q v_i(t)$, $\underset{n\to\infty}{lim} D^q w_i^n(t) = D^q w_i(t)$ $t \in (0,T]$. Thus by relations $(v_i^0 \leq v_i^1 \leq v_i^2 \leq \ldots)$, it follows that $v_i(t)$ and $w_i(t)$ satisfy the following monotone property

$v_i^0 \le v_i^1 \le \ldots \le v_i^n \le v_i \le w_i \le w_i^n \le \ldots \le w_i^1 \le w_i^0$,
$D^q v_i^0 \le D^q v_i^1 \le \ldots \le D^q v_i^n \le \ldots \le D^q w_i^n \le \ldots \le D^q w_i^1 \le D^q w_i^0$. Now, we prove that $v_i(t)$, $w_i(t)$ are respectively minimal and maximal solutions of initial value problem $(1.1)-(1.2)$. Since $f_i(i=1,2)$ is continuous then clearly the function $\sigma(\eta_i(t))$ is continuous and monotone nondecreasing in $v_i(t)$ implies that $\{\sigma(v_i^n(t))\}$ converges to $\sigma(v_i)(t)$, $t \in (0,\, T]$. Taking limit as $n \to \infty$ of $\{v_i^n(t)\}$ and using dominated convergence theorem, $v_i(t)$ satisfies the integral equation

$$v_i(t) = \Gamma(q)u_i^0 e_q(\lambda_2^i t) + \Gamma(q)(u_i^1 - \lambda_2^i u_i^0)\left[e_q(\lambda_2^i x) * e_q(\lambda_1^i x)\right](t) + \left[e_q(\lambda_2^i x * e_q(\lambda_1^i x)\sigma(v_i)(x)\right](t).$$

Thus $v_i(t)$ is an integral representation of the solution of IVP $(1.1)-(1.2)$. By the assumption of the function $f_i(i=1,2)$ and Lemma 2.7, it follows that $v_i(t)$ is a classical solution of IVP $(1.1)-(1.2)$. This proves that the lower sequence $\{v_i^n(t)\}$ converges to a solution $v_i(t)$ of IVP $(1.1)-(1.2)$. Similarly, we can prove that the upper sequence $\{w_i^n(t)\}$ converges to a solution $w_i(t)$ of IVP $(1.1)-(1.2)$ and satisfies the relation $v_i(t) \le w_i(t), i = 1,2, t \in (0,\, T]$. It follows that the relation
$v_i^0 \le v_i^1 \le v_i^2 \ldots \le v_i^n \le v_i \le w_i \le w_i^n \le \ldots \le w_i^1 \le w_i^0$,
holds as well as $v_i(t)$ and $w_i(t)$ are minimal and maximal solution of IVP $(1.1)-(1.2)$ on the sector Ω. Now we prove $v_i(t) = w_i(t), i = 1,2$, is unique solution of IVP $(1.1)-(1.2)$. It is sufficient to prove that $v_i(t) \ge w_i(t)$, $D^q v_i(t) \ge D^q w_i(t)\, t \in (0,\, T]$. For this, we consider $u_i(t) = v_i(t) - w_i(t)$. Then from IVP (1.1)-(1.2) and above hypothesis, we have

$$\begin{aligned}
(D^{2q}u_i)(t) + N_i(D^q u_i(t) + M_i u_i(t) &= (D^{2q}v_i)(t) - (D^{2q}w_i)(t) + \\
&\qquad N_i D^q v_i(t) - N_i D^q w_i(t) + M_i v_i(t) - M_i w_i(t), \\
&= f_i(t, v_1, v_2, D^q v_1, D^q v_2) - f_i(t, w_1, w_2, D^q w_1, D^q w_2) + \\
&\qquad N_i(D^q v_i - D^q w_i) + M_i(v_i - w_i), \\
&\ge -N_i(D^q v_i - D^q w_i) - M_i(v_i - w_i) + N_i(D^q v_i - D^q w_i) + \\
&\qquad M_i(v_i - w_i) \ge 0,\ t \in (0,\, T],
\end{aligned}$$

and $t^{1-q}u_i(t) = 0$, $t^{1-q}(D^q u_i)(t) = 0$. Then by Lemma 2.10, $u_i(t) \ge 0, \Rightarrow$ $v_i(t) \ge w_i(t)$, $D^q v_i(t) \ge D^q w_i(t) \quad t \in (0, T]$. Thus $u_i(t) = v_i(t) = w_i(t)$ is unique solution of IVP $(1.1)-(1.2)$. □

Now we prove uniqueness of solution of the IVP $(1.1)-(1.2)$.

THEOREM 3.2. *Assume that*

(i) v_i^0 *and* w_i^0 *in* C_{1-q}^q *are ordered lower and upper solutions of IVP* $(1.1)-(1.2)$

(ii) $f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_1) \in C(J \times \mathbb{R}^4, \mathbb{R})$ *is quasimonotone nondecreasing*

(iii) $f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_1)$ *satisfies both sided Lipschitz condition.*

Then the IVP $(1.1) - (1.2)$ *has unique solution in the sector* $[v_i^0, w_i^0]$.

Proof. Observe that

$$\begin{aligned}
-N_i(D^q u_i - D^q u_i^*) - M_i(u_i - u_i^*) &\leq f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_2) - \\
&\qquad f_i(t,\, u_1^*,\, u_2^*,\, D^q u_1^*,\, D^q u_2^*) \\
&\leq N_i(D^q u_i - D^q u_i^*) + M_i(u_i - u_i^*)
\end{aligned}$$

for $v_i^0 \leq u_i^* \leq u_i \leq w_i^0$ which follows from (2.3). Then the Theorem 3.1 implies that the problem $(1.1) - (1.2)$ has unique solution in sector $[v_i^0,\, w_i^0]$. $\square$

4. Conclusion

Existence and uniqueness of solutions of nonlinear system of initial value problems is obtained using monotone method coupled with lower and upper solutions.

Acknowledgement. Authors are very much thankful to the anonymous referee for careful reading of the paper and for their useful comments.

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